P 17 Domain decomposition methods for nonlinear transmission conditions (<u>O. Steinbach</u>, B. Wohlmuth) $\rightarrow$ IS, NS

The focus of this project in on the design and analysis of domain decomposition methods for time–dependent partial differential equations with nonlinear transmission conditions. As a model problem we consider the Richards equation modeling the saturated–unsaturated flow through a porous medium. By applying Kirchhoff transformations within non–overlapping subdomains we reformulate the nonlinear diffusion terms by means of nonlinear transmission conditions which are then discretized by using a mortar finite element domain decomposition method. Instead of some time–stepping scheme we use a space–time finite element method which allows for adaptive refinement and parallel solution strategies in space and time.

State of the art. Domain decomposition methods [5, 8] are a well established tool to couple different physical models, and to provide a unified framework for an efficient solution of the related discrete models. For the coupling of non–matching finite element meshes along interfaces, hybrid [6] and mortar [9] domain decomposition methods can be used, where the involved discrete stability conditions have to be considered with care. The finite element discretization of time–dependent partial differential equations is well established [7] but relies in most cases on the use of some time–stepping schemes. First results for more general space–time finite element methods are available, see [4] and the references given therein. Note that such an approach is advantageous when considering adaptivity as well as parallelization in space and time.

In particular in fluid mechanics domain decomposition methods can be used to couple different flow flow models, e.g., when modeling the saturated–unsaturated fluid flow through a porous medium, see, e.g., [1, 3] and the references given therein. Following [1], the Kirchhoff transformation is locally applied to reformulate the nonlinear diffusion terms. In combination with an implicit time discretization this results in a coupled system with nonlinear transmission conditions which can be incorporated in a weak formulation by using Lagrange multipliers [2]. The global nonlinear system can then be solved by applying a Newton method, and the linearized systems can be solved by some Dirichlet–Neumann domain decomposition methods [1].

In the ongoing PhD thesis project of M. Gsell we have applied the Kirchhoff transformation to the model problem of a stationary nonlinear diffusion equation, where the diffusion coefficient is different within non–overlapping subdomains. This results in a coupled problem of local Poisson equations, but with nonlinear transmission conditions. The mathematical and numerical analysis was done for both the continuous and discrete problem. For the discretization a primal hybrid formulation was used, and the implementation was done for model problems in three space dimensions.

Thesis project to be supervised by O. Steinbach. The aim of this thesis project is to formulate, to analyze, and to implement an efficient and stable domain decomposition method for the nonlinear transmission problem of the time–dependent Richards equation where the relative permeability may be different within non–overlapping subdomains. For the reformulation of the nonlinear diffusion terms we first apply local Kirchhoff transformations which results in simplified partial differential equations locally, but nonlinear transmission conditions occur. Instead of using time–stepping schemes for the discretization in time, we will apply a space–time finite element approach which allows to use general decompositions of the space–time domain into finite elements which are pentatops in the case of three–dimensional spatial domains. Since we will use mortar finite element methods for the discretization of the nonlinear transmission conditions we end up with a space–time finite element mortar domain decomposition method.

The main focus of this project will be on the numerical analysis of the proposed approach. This includes existence and uniqueness of solutions of the transformed system of partial differential equations with nonlinear transmission conditions, and the stability and error analysis of space–time finite element methods. The latter covers stable discretizations of the nonlinear transmission conditions by using mortar finite element methods to allow non–matching finite element meshes along the interfaces, and the stability analysis of the space–time finite element approach. This approach will be implemented in parallel to be able to handle three–dimensional spatial domains. Further topics of this project include the formulation and implementation of appropriate preconditioning strategies, and a posteriori error estimators to allow adaptive mesh refinement strategies. However, the numerical analysis of suitable parallel preconditioning strategies as well as the a posteriori error analysis will not be in the focus of the proposed thesis project. Instead these are possible topics for associated thesis projects as well as for further funding periods.

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