P 10 Optimal control approaches for optical flow (K. Bredies, M. Ulbrich) $\rightarrow$ NS, IS

This project investigates optimal control approaches for image sequence interpolation based on optical flow. The underlying transport equation, the optimization problem, and optimality conditions are investigated in suitable function space settings. Adjoint-based optimization methods that meet the needs of modern data terms and regularizations are developed.

State of the art. Using measurements of a continuous image sequence with intensity (brightness) $I : \Omega \times [0,T] \to \mathbb{R}_+$ where $\Omega \subset \mathbb{R}^d$ (e.g., $d = 2$, Ω rectangle), optical flow computations aim at determining the travel velocity $b(x,t)$ of the points $x \in \Omega$ for selected times $t \in [0,T]$. The underlying optical flow constraint is a transport equation that expresses the constancy of intensities along point trajectories: $I_t(x,t) + b(x,t) \cdot \nabla_x I(x,t) = 0$.
The analysis of this transport equation (existence, uniqueness, continuity or differentiability with respect to parameters) requires care, especially if b is nonsmooth [1, 10, 12, 22].
Typical optical flow approaches reconstruct b on a grid by inserting data-based approximations of I_t and $\nabla_x I$ into the optical flow equation. Further constraints or regularizations are required to make the problem well-posed and numerous proposals for this exist, e.g. [8, 9, 23, 27, 25]. Many optical flow computations follow Horn and Schunck [15] and minimize the sum of a data term, involving the above spatio-temporal derivatives of I, and a regularization term.
Similar to [5, 6, 10, 11], this project will obtain an approximation of the *whole* velocity field on $Q := \Omega \times (0,T)$ from data I_j for $I(t_j)$ at times $t_j \in [0,T]$, $0 \le j \le N$. To this end, an optimal control problem (P) of the following form is solved:

$$\min_{b,I} \sum_{j=1}^N D(I(t_j), I_j) + R(b) \quad \text{s.t.} \quad I_t + b \cdot \nabla_x I = 0 \ \text{ in } Q, \quad I(0) = I_0 \quad (\text{+ other constr.}).$$

Here, D is a data (regression) term, e.g., $D(I(t_j), I_j) = \|I(t_j) - I_j\|^2_{L^2(\Omega)}/2$ and R is a regularization term. D or R sometimes include structured nonsmoothness (such as TV- or L^1-terms) or smoothed versions of them. Prior knowledge on b and I can either be expressed as constraints or via regularization terms. There also exist promising connections between optical flow and optimal transportation [16, 20] that will be explored.

Thesis project to be supervised by Michael Ulbrich. This project investigates theoretical and numerical aspects of image sequence interpolation from the perspective of PDE-constrained optimal control with optical flow constraints. For the development of a suitable function space setting, a particular challenge consists in devising appropriate, weak assumptions on b such that the optical flow equation possesses solutions I that are stable with respect to b. Based on work on transport equations with nonsmooth coefficients [1, 7, 12, 13] such existence and stability results shall be derived and the existence of solutions to the regularized inverse problem will be studied. In this context, suitable regularization terms and constraints shall be investigated. A starting point is, e.g., [10], where the case $b \in L^2(0,T; H_0^3(\Omega)^2)$ and $\operatorname{div} b = 0$ is considered. For applying optimization theory and methods, the differentiability properties of the optical flow equation have to be studied and a consistent solution concept for the forward and adjoint equations is required. Taking into account the challenges of shock movements [22], the optimality system will be derived

and results on the differentiability of the reduced objective function will be developed.
We will initially use a problem formulation (P) similar to [10] and [5, 6]. The problem complexity will then gradually be extended, with suitable adjustments to the respective studies. For instance, the constraint $\operatorname{div} b = 0$ in [10, 11] is convenient, but not always justifiable and Sobolev regularizations usually are too strong. Instead, robust regularizers (e.g., Huber-type, L^1, TV) have proven advantageous [8, 26, 27]. They introduce additional nonlinearity and nonsmoothness that have to be addressed. Other regularizations, e.g., using fundamental matrices [25, 23] or subspace constraints [14] would also be interesting to study in our setting. Algorithmically, we can start from our preliminary implementation of an adjoint-based first order method similar to [10] and a basic coarse-to-fine nested iteration. We will study and implement accelerated first-order methods, building on earlier experience [18, 21] and we also will work on improvements of the multilevel approach. Nonsmooth regularization terms shall be handled by splitting methods, where variants of proximal splitting [2, 4, 21] and of primal-dual [17, 19, 24] methods will be investigated. Jointly with K. Bredies, we also will explore promising connections to optimal transport [16, 20] (cf. Further topics). The Middlebury benchmark [3] will be used to test the developed algorithms.

Further topics. This IGDK-project started in 2014 and P. Jarde (PhD student, TUM) will continue to work on it. During his visit (3–5/15) to Graz, he and K. Bredies started joint research (which will be continued) on image interpolation strategies where the optical flow constraint might be relaxed. It bases on the observation that latter can be written as the inhomogeneous conservation law $I_t + \operatorname{div}_x(Ib) = I \operatorname{div}_x b$, motivating optimal transport approaches. Assuming that $\operatorname{div}_x b = 0$, the image interpolation problem can then be solved by convex minimization with respect to I and $w = Ib$, [16, 20]. We expect a similar behavior if $\operatorname{div}_x b$ is known and will study how this can be incorporated into the optimization framework. One approach will be to penalize $\operatorname{div}_x b$ as well as to introduce the bilinear constraint $w = Ib$ or a relaxation of it. Without the latter, the optimization problem can be made convex, in particular giving rise to new algorithmic strategies that exploit this specific structure.

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