P 8 Parameter estimation for the Bloch equation from linear projection data and applications in MR fingerprinting (K. Bredies, M. Fornasier) $\rightarrow$ NS, IS

The image acquisition process in magnetic resonance imaging (MRI) bases on the manipulation of hydrogen nuclei magnetic moments by magnetic fields. These are essentially described by the so-called net magnetization vector which is governed by the Bloch equation. This equation depends on several parameters which, in common MRI, have an indirect influence on the reconstructed images. Recently, with a technique called *MR fingerprinting* (MRF), first attempts were made to recover all parameters directly and simultaneously using a unified acquisition strategy. Mathematically, these were determined by template matching for a set of precomputed solutions of the Bloch equation. In this project, the parameters are directly recovered from the Bloch equation by solving the associated parameter estimation problem.

State of the art. The development of acquisition and reconstruction techniques in MRI is a highly active interdisciplinary field of research. Currently, various qualitative imaging techniques such as T_1 and T_2 *weighted* imaging are well-established in terms of data acquisition and acceleration in terms of undersampling and "compressed sensing" [11, 9, 10]. In contrast, T_1 and T_2 *mapping* approaches require dedicated acquisition strategies tailored towards the measurement of specific parameters [6, 4], also in the undersampling case [7, 8]. With the recently introduced MRI fingerprinting approach [12], the simultaneous recovery of the relaxation times and spin density was performed using a uniform and simple acquisition strategy. However, this approach massively relies on large collection of data, even in the case of undersampling. Recently, more appropriate compressed sensing reconstructions have been incorporated in the MRF process [5], however, a full modeling and undersampling reconstruction directly based on the Bloch equations has not been performed so far.

Thesis project to be supervised by Kristian Bredies. This project will be concerned with the study of the parameter identification problems for the Bloch equation in the context of MRI. The mathematical problem is to determine the spatially-varying relaxation parameters T_1, T_2, spin density ρ and components of the field inhomogeneity δB in the Bloch equation

$$\begin{cases} \dfrac{\partial M}{\partial t} = \gamma M \times (B + \delta B) - \begin{pmatrix} \frac{M_x}{T_2} \\ \frac{M_y}{T_2} \\ \frac{M_z - \rho}{T_1} \end{pmatrix}, \quad M(0) = (0, 0, \rho) \end{cases} \tag{1}$$

for the net magnetization vector field $M = (M_x, M_y, M_z)$ and a given spatio-temporal dependent magnetic flux field B (which will be influenced during the acquisition process). The data available in order to solve the problem will correspond to the voltage which is induced in the MR measurement coils reading as, up to factors,

$$s_i(t) = \int_{\mathbb{R}^3} C_i \cdot \frac{\partial M}{\partial t}(t, x) \, \mathrm{d}x \tag{2}$$

where $C_i : \mathbb{R}^3 \to \mathbb{R}^3$ is the sensitivity vector field associated with the i-th receive coil. Usually, s_i is available on several data-acquisition time intervals $[t_0, t_1]$, depending on the utilized imaging sequence. Then, the problem of determining the MR parameters is finding T_1, T_2, ρ and δB subject to (1) and (2) in a $\Omega \subset \mathbb{R}^3$, which is a parameter identification problem for the Bloch equation subject to linear constraints. It will be studied in terms of solvability of associated optimization problems. As undersampling will be involved and the data is assumed to be incomplete, the basic approach will be to minimize a suitable "sparsifying" penalty subject to the measurement constraints. As the *total generalized variation* (TGV) turned out to be successful in the context of undersampling MRI [2, 9], the problem

$$\min_{T_1,T_2,\rho,\delta B} \ \mathrm{TGV}_\alpha^k(T_1, T_2, \rho, \delta B) \quad \text{subject to} \quad (1) \text{ and } (2) \tag{3}$$

is considered, where TGV_α^k is the k-th order total generalized variation (typically, $k = 2$) with weights α measuring simultaneously, in a suitable vectorial sense, the regularity of all of its arguments. As this problem is non-convex, non-smooth and incorporates linear constraints, it may numerically be solved, for instance, by [1].

Furthermore, convex relaxation approaches will be studied within the project, as these turn out to possess desirable properties with respect to global minimization. One approach will be to "lift" the parameters in (1) responsible for the non-convexity, for instance, neglecting δB, the quantities T_1 and T_2. Such a lifting basically represents the parameters T_1 and T_2 by a delta peak $\delta_{(T_1,T_2)}$. Denoting P as the parameter space, a convex relaxation approach would then recover a positive Radon measure μ on $\Omega \times P$ whose projection onto Ω is absolutely continuous with respect to the Lebesgue measure. For this approach, a lifted version of (1) as well as TGV has to be found and analyzed such that (3) becomes convex with respect to μ. As the concentration of the measure with respect to parameter space cannot be enforced in a convex manner, a suitable "sparsifying" penalty such as the Radon norm will be incorporated. Consequently, the optimization problem (3) becomes a problem in Radon space which can, analytically and numerically be treated by the techniques presented, for instance, in [3].

The goals of the project can be summarized to be:

(a) To study, analytically and numerically, the parameter estimation problem for the Bloch equation (3) in function space for realistic situations in undersampling MRI,

(b) to develop a convex relaxation framework in order to support numerical minimization algorithms in finding more global solutions.

Further topics. The proposed parameter estimation framework for the Bloch equations has natural extensions regarding the incorporation of physical effects which are not covered by the Bloch equation. For instance, diffusion processes may also be measured leading to *diffusion MRI*. A possibility to model these processes is given by the Bloch-Torrey equation [13] which constitutes a linear diffusion equation with diffusion tensor as a parameter to be identified. While the framework essentially remains the same, additional effort is required in order to incorporate such equations which are now partial differential equations.

Bibliography

[1] M. Artina, M. Fornasier, and F. Solombrino. Linearly constrained nonsmooth and nonconvex minimization. *SIAM Journal on Optimization*, 23(3):1904–1937, 2013.

[2] K. Bredies, K. Kunisch, and T. Pock. Total generalized variation. *SIAM Journal on Imaging Sciences*, 3(3):492–526, 2010.

[3] K. Bredies and H. K. Pikkarainen. Inverse problems in spaces of measures. *ESAIM: Control, Optimisation and Calculus of Variations*, 19(1):190–218, 2013.

[4] F. Cao, O. Commowick, E. Bannier, and C. Barillot. Simultaneous estimation of T_1, T_2 and B_1 maps from relaxometry MR sequences. In *Proceedings of MICCAI Workshop on Intelligent Imaging Linking MR Acquisition and Processing*, pp. 16–23. 2014.

[5] M. Davies, G. Puy, P. Vandergheynst, and Y. Wiaux. A compressed sensing framework for magnetic resonance fingerprinting. *SIAM Journal on Imaging Sciences*, 7(4):2623–2656, 2014.

[6] S. C. L. Deoni, B. K. Rutt, and T. M. Peters. Rapid combined T_1 and T_2 mapping using gradient recalled acquisition in the steady state. *Magnetic Resonance in Medicine*, 49:515–526, 2003.

[7] M. Doneva, P. Börnert, H. Eggers, C. Stehning, J. Sénégas, and A. Mertins. Compressed sensing reconstruction for magnetic resonance parameter mapping. *Magnetic Resonance in Medicine*, 64:1114–1120, 2010.

[8] C. Huang, C. G. Graff, E. W. Clarkson, A. Bilgin, and M. I. Altbach. T_2 mapping from highly undersampled data by reconstruction of principal component coefficient maps using compressed sensing. *Magnetic Resonance in Medicine*, 67:1355–1366, 2012.

[9] F. Knoll, K. Bredies, T. Pock, and R. Stollberger. Second order total generalized variation (TGV) for MRI. *Magnetic Resonance in Medicine*, 65(2):480–491, 2011.

[10] F. Knoll, C. Clason, K. Bredies, M. Uecker, and R. Stollberger. Parallel imaging with nonlinear reconstruction using variational penalties. *Magnetic Resonance in Medicine*, 67(1):34–41, 2012.

[11] M. Lustig, D. Donoho, and J. M. Pauly. Sparse MRI: the application of compressed sensing for rapid MR imaging. *Magnetic Resonance in Medicine*, 58:1182–1195, 2007.

[12] D. Ma, V. Gulani, N. Seiberlich, K. Liu, J. L. Sunshine, J. L. Duerk, and M. A. Griswold. Magnetic resonance fingerprinting. *Nature*, 495:187–192, 2013.

[13] H. Torrey. Bloch equations with diffusion terms. *Physical Review*, 104(3):563, 1956.