# P 2 Complexity and convergence of multilevel optimization approaches for TV-based denoising (F. Bornemann, K. Bredies) $\rightarrow$ AO, NS, IS

Variational problems in mathematical image processing such as total-variation (TV) denoising, for instance, are often non-smooth but spatially structured. This spatial structure is typically revealed by optimality conditions which formally constitute a nonlinear partial differential equation. For such equations, one would like to employ multilevel method which yield, in case of sufficient smoothness, optimal complexity and convergence rates. However, the underlying equations are non-smooth in the TV-denoising case and such techniques are not directly applicable such that designing efficient multilevel schemes with desirable convergence properties is a challenge. This project aims at approaching this problem by developing and analyzing new optimization-based multilevel schemes with the prototypical example of TV denoising in focus.

**State of the art.** There is a vast amount of literature available on numerical algorithms for the solution of variational imaging problem, in particular for total-variation denoising [16]. Among those, first-order algorithms are constituting the state of the art [13, 10, 7, 6], where also optimal rates in terms of worst-case complexity can be achieved [2, 10]. However, regarding multilevel approaches in conjunction with total-variation minimization, only few works are available [11, 12, 9, 8]. Likewise, concerning convergence of discrete solutions towards the continuous one, finite-difference approximations [17] as well as finite-element approximations [1, 15] have been analyzed.

**Thesis project to be supervised by Kristian Bredies.** The thesis project is concerned with the analysis of the complexity and convergence of additive multilevel schemes for the numerical solution of variational imaging problems. The starting point will be the study, for $\Omega = (0,1)^2$, of numerical approximations of the solution to the total-variation denoising problem

$$\min_{u \in L^2(\Omega)} \frac{1}{2} \int_\Omega (u-f)^2 \, \mathrm{d}x + \lambda \int_\Omega \mathrm{d}|Du| \tag{1}$$

by a coarse-to-fine or "one-way" multilevel scheme inspired by [9]. For $\Omega_k$ a regular finite-difference grid after $k$ refinements, i.e., $\Omega_k = \{(i2^{-k}, j2^{-k}) : 1 \leq i, j \leq 2^k\}$, the discrete problem

$$\min_{u^k : \Omega_k \to \mathbb{R}} F_k(u^k), \qquad F_k(u^k) = 2^{-2k} \sum_{i=1}^{2^k} \sum_{j=1}^{2^k} \frac{1}{2}(u_{ij}^k - f_{ij}^k)^2 + 2^k \lambda |(\nabla u^k)_{ij}| \tag{2}$$

where $\nabla$ represents a finite-difference operator is considered. This can be solved, for instance, with primal-dual algorithms. With the associated primal-dual gap function, optimality of (2) may be assessed for the iterates and rates in terms of iteration numbers are known. In particular, one can switch to a finer discretization, i.e., $k \to k+1$, once the iterate is optimal up to a $\varepsilon_k > 0$ and use a prolongation of the last iterate as initial value. The project now aims at coupling this technique with the convergence behavior as $k \to \infty$, i.e., as the discretization becomes finer [17]. Here, appropriate restriction strategies for the data $f$ as well as prolongation strategies for the discrete approximate solutions $u^k$ have to be taken into account. The goal is to derive the complexity to approach the continuous solution within a certain tolerance for suitable primal-dual algorithms, assuming that each iteration step is polynomial in the number of unknowns.

The second stage of the project will deal with the incorporation of multigrid approaches into this framework as well as extending the complexity considerations in terms of the number of elementary computational operations. Here, the starting points are preconditioned Douglas-Rachford iterations [7, 6] which ask for the approximate discrete solution of the equation

$$\begin{cases} -\mu\Delta u + \lambda u = b \quad \text{in } \Omega, \quad \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3}$$

The project involves incorporating classical multigrid techniques for which a reduction of the error in (3) can be achieved in $\mathcal{O}(N)$ operations independent from the discretization level, but also aims at employing additive multilevel approaches (i.e., based on stable generating sets or 'frames') [4, 14, 3] which are known to be particularly efficient. In particular, the project will focus on multilevel formulations of the optimization problem (2) of the form

$$\min_{v^0,\ldots,v^k} \max_{k' \in \{0,\ldots,k\}} F_{k'}(v^{k'}) \quad \text{subject to} \quad v^{k'} = R_{k'} v^{k'+1}, \quad k' = 0,\ldots,k-1 \tag{4}$$

with suitable $f^{k'}$, restriction operators $R_{k'}$ and possibly level-dependent $\nabla$. These are aimed to be designed such that the highest-level solution $v^k$ is the solution of (2) and the lower-level solutions $v^{k'}$, $k' = 0,\ldots,k-1$ provide approximations to the respective solutions of (2). Such a condition can, for instance, be fulfilled by requiring that $F_{k'}(R_{k'} v^{k'+1}) \leq F_{k'+1}(v^{k'+1})$ for $k' = 0,\ldots,k-1$. This formulation will then be equivalent to (2) but affect numerical algorithms, in particular preconditioned Douglas-Rachford iterations. It will then be studied which choices of preconditioners lead to an inherent multilevel structure comparable to additive multilevel approaches and based on this, complexity and convergence will be estimated.

The goals of the project can be summarized as follows:

(a) To analyze the complexity and convergence of basic coarse-to-fine approaches for total-variation denoising in conjunction with primal-dual algorithms,

(b) to establish and analyze an additive multilevel optimization framework for total-variation denoising which allows for simultaneous optimization on all levels.

**Further topics.** Having successfully analyzed the case of TV denoising allows to go one step further and incorporate higher-order regularization approaches. Particularly interesting is the case where TV is replaced by the *total generalized variation* (TGV), which significantly reduces the staircasing effect in TV denoising while maintaining the possibility to reconstruct discontinuities [5]. First steps towards multilevel techniques for higher-order TV have already been presented in [8], serving as a starting point for incorporating TGV into the framework.

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