Interface control of multi-phase fluid flow: Analytical and numerical aspects

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We consider the following optimal control problem for the interface in a two-dimensional multi-phase fluid problem: for a fixed $\tilde{\rho}$, find $\boldsymbol{y}^*, \boldsymbol{u}^* : \Omega_T \to \mathbb{R}^2$, and $\rho^* : \Omega_T \to \mathbb{R}$, such that

$$J(\rho^*, \boldsymbol{u}^*) = \min \int_0^T \left\{ \beta \mathcal{H}^1(S_\rho) + \frac{\lambda}{2} \int_\Omega |\rho - \tilde{\rho}|^2 \, \mathrm{d}\boldsymbol{x} + \frac{\alpha}{2} \int_\Omega |\boldsymbol{u}|^2 \, \mathrm{d}\boldsymbol{x} \right\} \mathrm{d}t$$

subject to

$$\rho \boldsymbol{y}_t + \rho [\boldsymbol{y} \cdot \nabla] \boldsymbol{y} - \operatorname{div}(\mu(\rho) \nabla \boldsymbol{y}) + \nabla p = \rho \boldsymbol{u}, \tag{1a}$$

$$\rho_t + [\boldsymbol{y} \cdot \nabla] \rho = 0, \tag{1b}$$

$$\operatorname{div} \boldsymbol{y} = 0, \tag{1c}$$

together with $\rho(0,.) = \rho_0$, $\boldsymbol{y}(0,.) = \boldsymbol{y}_0$, and $\boldsymbol{y} = \boldsymbol{0}$ on $(0,T] \times \partial\Omega$. Here, $\mathcal{H}^1(S_\rho)$ is the interfacial length of the interface between the two faces of the fluid mixture.

The problem is motivated in order to control the interface of a two-phase fluid and to avoid oscillatory effects on the interface, as well as to handle topological changes. A possible example is the control of aluminium production via electrolysis.

We discuss technical problems, which lead to an approximation by a phase-field approximation with an parameter δ and the presents of artificial diffusion of order ε in the mass equation (1b). For fixed parameters $\delta, \varepsilon > 0$, we show existence of an optimum of the regularized problem, and we derive first order necessary optimality conditions.

We use an unconditionally stable fully practical discrete scheme which is based on low order finite elements in order to define a discrete optimization problem corresponding to the continuous one above. For the discrete problem and positive paremters $\delta, \varepsilon > 0$, we show existence of a discrete optimal solution, derive discrete first order necessary optimality conditions and show convergence of corresponding iterates to solutions of the limiting optimality conditions for vanishing discretization parameters (up to subsequences).

Computational studies gives hints on the coupling between $\delta, \varepsilon > 0$, as well as the validation of the model including interface motion, topological changes. Moreover, we present experiments showing a different qualitative behavior for $\beta = 0$ and $\beta > 0$, respectively, as well as $\lambda = 0$ and $\lambda > 0$, respectively.

This is a joint work with Ľubomír Baňas and Andreas Prohl.